

# Poincaré inequality and the uniqueness of solutions for the heat equation associated with subelliptic diffusion operators

Bumsik Kim

Department of Mathematics, Purdue University  
West Lafayette, IN, USA

## Abstract

In this paper we study global Poincaré inequalities on balls in a large class of sub-Riemannian manifolds satisfying the generalized curvature dimension inequality introduced by F.Baudoin and N.Garofalo. As a corollary, we prove the uniqueness of solutions for the subelliptic heat equation. Our results apply in particular to CR Sasakian manifolds with Tanaka-Webster-Ricci curvature bounded from below and Carnot groups of step two.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Preliminaries</b>	<b>4</b>
<b>3</b>	<b>Poincaré inequality on the ball</b>	<b>7</b>
3.1	Lower bound of the Dirichlet heat kernel on the ball . . . . .	7
3.2	proof of Theorem 1.1 . . . . .	10
3.3	Sobolev inequality and $L^p$ mean value estimate . . . . .	12
<b>4</b>	<b>Uniqueness of the positive solution</b>	<b>16</b>
4.1	Minimality of the heat semigroup for positive solutions . . . . .	16
4.2	proof of Theorem 1.2 . . . . .	17
<b>5</b>	<b>Uniqueness of <math>L^p</math> solution</b>	<b>18</b>
5.1	proof of Theorem 1.3, $p > 1$ . . . . .	18
5.2	Hamilton's inequality . . . . .	20
5.3	proof of Theorem 1.3, $p = 1$ . . . . .	21

# 1 Introduction

In this paper,  $\mathbb{M}$  is a  $C^\infty$  connected finite dimensional manifold endowed with a smooth measure  $\mu$ . Let  $L$  be a second-order diffusion operator on  $\mathbb{M}$ , which is symmetric, non-positive, locally subelliptic in the sense of [17], [25], [18], with  $L1 = 0$ .

By the subellipticity of  $L$ , there is an intrinsic distance  $d(x, y)$  associated to  $L$  on  $\mathbb{M}$  ([17], [25]). If we denote  $\Gamma(f) := \Gamma(f, f)$  (the Bakry-Émery's *carré du champ* of  $L$ , see [3]) by the quadratic differential form  $\Gamma(f, g) = \frac{1}{2}(L(fg) - fLg - gLf)$ ,  $f, g \in C^\infty(\mathbb{M})$ , the distance  $d(x, y)$  is defined via the notion of subunit curves with respect to the *length of the gradient*,  $\sqrt{\Gamma(f)}$ . Throughout this paper, we assume that our sub-Riemannian metric space  $(M, d)$  is complete.

This class of operators includes Hörmander type operators of order  $k$  (or sum of squares of vector fields satisfying Hörmander condition), Grushin's operator and sub-Laplacians on Sasakian CR-manifolds (see [25], [7]). For domains whose diameters are bounded in terms of the subellipticity constants, the Poincaré inequality was proved in [24] for the Hörmander type operator, and the general subelliptic Poincaré inequality can be found in Kusuoka and Stroock's work [27] through their probabilistic methods.

In general, the global Poincaré inequality could fail if the domain is not bounded ([24], even in the Riemannian case). On the Riemannian manifold with Ricci curvature bounded from below by  $-K < 0$ , the global Poincaré inequality is proved by Buser [12]:

$$\int_{B(x,r)} |f - f_{B(x,r)}|^2 d\mu \leq C_1 r^2 e^{C_2 \sqrt{K}r} \int_{B(x,r)} |\nabla f|^2 d\mu, \quad \forall f \in C^\infty(\mathbb{M}), r > 0. \quad (1.1)$$

This was done through geodesic arguments equipped with geometric tools such as the Bishop-Gromov comparison theorem.

In our subelliptic framework, due to the lack of several properties such as ellipticity of  $L$  or smoothness of the distance function, we cannot use many of the tools for Riemannian manifolds. As a recent breakthrough, *the generalized curvature dimension inequality*  $CD(\rho_1, \rho_2, \kappa, d)$  of Baudoin and Garofalo [7] was introduced to specify a certain subelliptic curvature condition ([7], [4], [5], [6], [8], [9], [10]). (Definition 2.1 in the present paper)

For example, in [7] it was shown that if the lower bound of Tanaka-Webster-Ricci tensor on a CR Sasakian manifold of codimension 1 is given by  $\rho_1 \in \mathbb{R}$ , the sub-Laplacian of  $\mathbb{M}$  satisfies  $CD(\rho_1, d/4, 1, d)$  where the real dimension of the distribution of the CR manifold is  $d$ .

If  $\rho_1$  is non-negative (an analogue of the non-negative lower bound of Ricci tensor in the Riemannian manifold), some types of Poincaré inequalities were discussed in [5], [10], including Buser's Poincaré inequality with  $K = 0$ .

Our purpose in the present paper is to prove the Buser's global Poincaré inequality in the subelliptic framework with the condition  $CD(-K, \rho_2, \kappa, d)$ ,  $K > 0$ :

**Theorem 1.1** (Poincaré inequality). *If  $\mathbb{M}$  satisfies  $CD(-K, \rho_2, \kappa, d)$  with  $K > 0$ , for any*

$r > 0$ ,  $x_0 \in \mathbb{M}$  and  $f \in C^\infty(\mathbb{M})$ ,

$$\int_{B(x_0, r)} |f(x) - f_{B(x_0, r)}|^2 d\mu(x) \leq C_{p1} r^2 e^{C_{p2} K r^2} \int_{B(x_0, r)} \Gamma(f) d\mu, \quad (1.2)$$

where  $C_{p1}, C_{p2} > 0$  depend on  $\rho_2, \kappa, d$ .

Once the global subelliptic Poincaré inequalities and the volume doubling property are assumed, one can obtain Sobolev inequalities through [36], and we can run Moser's iteration ([31],[32]) to recover various properties such as the uniqueness of the solution for the heat equation through the mean value estimates of subsolution of  $L$ -Laplace equation. Note that two sided heat kernel bounds and parabolic Harnack inequality for  $P_t f$  were already obtained through Li-Yau type inequalities and heat kernel methods in [7],[6]. However this Harnack inequality does not directly yield the uniqueness of positive solutions in contrast to Li-Yau's [29].

We consider the following heat equation associated with  $L$ .

$$\left( \frac{\partial}{\partial t} - L \right) u(x, t) = 0 \quad (1.3)$$

Adapting the ideas of [14],[30], we conclude our main corollaries, the uniqueness of the solutions for the subelliptic heat equation.

**Theorem 1.2** (Uniqueness of positive solutions). *Assume that  $\mathbb{M}$  satisfies the generalized curvature-dimension inequality  $CD(-K, \rho_2, \kappa, d)$ ,  $K > 0$  with respect to  $L$ . For any non-negative solution  $u \in C(\mathbb{M} \times [0, \infty))$  of (1.3),  $u$  is uniquely determined by the initial data  $u(\cdot, 0) = f$  and  $u(x, t) \equiv P_t f(x)$ , where  $P_t$  is the heat semigroup generated by  $L$ .*

**Theorem 1.3** (Uniqueness of  $L^p$  solutions). *For  $1 < p < \infty$ ,  $L^p$  solution  $u(x, t)$  of (1.3) defined on  $\mathbb{M} \times (0, \infty)$  is uniquely determined by the initial data  $f \in L^p(\mathbb{M})$ , i.e.  $u(\cdot, t) \in L^p(\mathbb{M})$  for any  $t > 0$  and  $u \xrightarrow{L^p} f$  as  $t \rightarrow 0$ .*

*If  $CD(-K, \rho_2, \kappa, d)$ ,  $K > 0$  is satisfied, we have an unique  $L^1$  solution  $u(x, t)$  of (1.3) with the initial data  $u \xrightarrow{L^1} f \in L^1(\mathbb{M})$  as  $t \rightarrow 0$ .*

Theorem 1.2 improves Li-Yau/Harnack inequality for the subelliptic  $L$  which is proved for  $P_t f$  in [7],[6]. (Find notations in the following section. See Remark 2.5 for  $\mathcal{A}_\epsilon$ .)

**Corollary 1.4** (Li-Yau type inequality, [7]). *Assume  $CD(\rho_1, \rho_2, \kappa, d)$ ,  $\rho_1 \in \mathbb{R}$ . Any non-negative solution  $u(x, t) \in \mathcal{A}_\epsilon$  of (1.3) satisfies the following inequality:*

$$\begin{aligned} & \Gamma(\ln u) + \frac{2\rho_2}{3} t \Gamma^Z(\ln u) - \left( 1 + \frac{3\kappa}{2\rho_2} - \frac{2\rho_1}{3} t \right) \frac{Lu}{u} \\ & \leq \frac{d\rho_1^2}{6} t - \frac{d\rho_1}{2} \left( 1 + \frac{3\kappa}{2\rho_2} \right) + \frac{d}{2t} \left( 1 + \frac{3\kappa}{2\rho_2} \right)^2. \end{aligned}$$

**Corollary 1.5** (Harnack inequality, [6],[7]). *Assume  $CD(-K, \rho_2, \kappa, d)$ ,  $K \geq 0$ . Then any non-negative solution  $u \not\equiv 0$  of (1.3) satisfies for  $x, y \in \mathbb{M}$ ,  $s < t \in \mathbb{R}_+$ ,*

$$\frac{u(x, s)}{u(y, t)} \leq \left( \frac{t}{s} \right)^{\frac{D}{2}} \exp \left( \frac{dK}{4} (t - s) + \frac{K}{12} d(x, y)^2 + \frac{d(x, y)^2}{4(t - s)} \left( 1 + \frac{3\kappa}{2\rho_2} \right) \right).$$

## 2 Preliminaries

Our subelliptic operator  $L$  is described as follows:  $L$  is a second-order diffusion operator with real  $C^\infty$  coefficients on  $\mathbb{M}$ . There exists a neighborhood  $U$  of  $x \in \mathbb{M}$  and a constant  $C > 0$ , such that for any  $f \in C_0^\infty(U)$ ,

$$\|f\|_\epsilon^2 \leq C(|\langle f, Lf \rangle| + \|f\|_2^2), \quad (2.4)$$

where  $\|f\|_\epsilon = (\int |\hat{u}(\xi)|^2 (1 + |\xi|^2)^\epsilon d\xi)^{1/2}$  is the Sobolev norm of order  $0 < \epsilon < 1$ , and  $\langle \cdot, \cdot \rangle, \|\cdot\|_2$  are respectively the inner product and the norm of  $L^2(\mathbb{M}, \mu)$ . Also  $L$  is symmetric, non-positive and has zero order term, i.e.:

$$\int_{\mathbb{M}} f L g d\mu = \int_{\mathbb{M}} g L f d\mu, \quad \int_{\mathbb{M}} f L f d\mu \leq 0, \quad L1 = 0,$$

for every  $f, g \in C_0^\infty(\mathbb{M})$ .

The intrinsic sub-Riemannian metric associated with  $L$  is defined by the minimal length of subunit curve:

$$d(x, y) = \inf \left\{ T \mid \exists \text{ Lipschitz } \gamma : [0, T] \rightarrow \mathbb{M}, \gamma(0) = x, \gamma(T) = y, \right. \\ \left. \left| \frac{d}{dt} f(\gamma(t)) \right| \leq \sqrt{\Gamma f(\gamma(t))}, \forall f \in C^\infty(\mathbb{M}), \text{ almost every } t \in [0, T] \right\},$$

where  $\Gamma(f, g) = \frac{1}{2}(L(fg) - fLg - gLf)$ ,  $\Gamma(f) = \Gamma(f, f)$ . We assume that the metric space  $(\mathbb{M}, d)$  is complete.

Note that this class of operators strictly includes Hörmander's type operators  $-\sum_{i=1}^m X_i^* X_i$  ( $X_i$ 's are the smooth vector fields satisfying Hörmander condition of order  $k$  on  $\mathbb{M}$ , and  $X_i^*$  is the formal adjoint of  $X_i$  in  $L^2(\mathbb{M}, \mu)$ ). Following Strichartz [38], the completeness assumption of  $(\mathbb{M}, d)$  yields that  $L$  is essentially self-adjoint on  $C_0^\infty(\mathbb{M})$ . So we can denote by  $L$  the unique self-adjoint extension (the Friedrichs extension) of  $L$  in  $L^2(\mathbb{M}, \mu)$ . Maximum principle([11]) and Hörmander's hypoellipticity of  $L$  are well-known. See [17],[24],[7],[33] for more properties of  $L$ .

We follow the steps in [7] to introduce the curvature assumption on our subelliptic framework. In addition to  $\Gamma$ , we assume that  $\mathbb{M}$  is endowed with another smooth symmetric bilinear differential form, indicated with  $\Gamma^Z$ , satisfying for  $f, g \in C^\infty(\mathbb{M})$

$$\Gamma^Z(fg, h) = f\Gamma^Z(g, h) + g\Gamma^Z(f, h),$$

and  $\Gamma^Z(f) = \Gamma^Z(f, f) \geq 0$ .

We make the following assumptions that will be in force throughout the paper:

(H.1) There exists an increasing sequence  $h_k \in C_0^\infty(\mathbb{M})$  such that  $h_k \nearrow 1$  on  $\mathbb{M}$ , and

$$\|\Gamma(h_k)\|_\infty + \|\Gamma^Z(h_k)\|_\infty \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

(H.2) For any  $f \in C^\infty(\mathbb{M})$  one has

$$\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f)).$$

(H.3) For every  $t \geq 0$ ,  $P_t 1 = 1$  and for every  $f \in C_0^\infty(\mathbb{M})$  and  $T \geq 0$ , one has

$$\sup_{t \in [0, T]} \|\Gamma(P_t f)\|_\infty + \|\Gamma^Z(P_t f)\|_\infty < +\infty,$$

where  $P_t$  is the heat semigroup generated by  $L$ .

(Details about the assumptions are discussed in [7]) The assumption (H.1) is implied by the completeness of the metric space. In the sub-Riemannian geometries covered by the present work, the assumption (H.2) means that the torsion of the sub-Riemannian connection is vertical (for instance, Sasakian condition of CR manifolds). Removing this assumption in certain cases is discussed in [9]. Assumption (H.3) is necessary to rigorously justify the Bakry-Émery type arguments. It is a consequence of the generalized curvature dimension inequality below in many examples (see [7]).

In addition to  $\Gamma$  and  $\Gamma^Z$ , we denote the following second order differential bilinear forms: for any  $f, g \in C^\infty(\mathbb{M})$ ,

$$\Gamma_2(f, g) = \frac{1}{2} [L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf)], \quad (2.5)$$

$$\Gamma_2^Z(f, g) = \frac{1}{2} [L\Gamma^Z(f, g) - \Gamma^Z(f, Lg) - \Gamma^Z(g, Lf)]. \quad (2.6)$$

As for  $\Gamma$  and  $\Gamma^Z$ , we denote  $\Gamma_2(f) = \Gamma_2(f, f)$ ,  $\Gamma_2^Z(f) = \Gamma_2^Z(f, f)$ .

The following curvature dimension condition was introduced in [7].

**Definition 2.1** ([7], generalized curvature dimension inequality). *We say that  $L$  satisfies the generalized curvature dimension inequality  $CD(\rho_1, \rho_2, \kappa, d)$  on  $\mathbb{M}$  if there exist constants  $\rho_1 \in \mathbb{R}$ ,  $\rho_2 > 0$ ,  $\kappa \geq 0$ , and  $0 < d < \infty$  such that the inequality*

$$\Gamma_2(f) + \nu \Gamma_2^Z(f) \geq \frac{1}{d} (Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right) \Gamma(f) + \rho_2 \Gamma^Z(f)$$

*holds for every  $f \in C^\infty(\mathbb{M})$  and every  $\nu > 0$ .*

The inequality  $CD(\rho_1, \rho_2, \kappa, d)$  turns out to be equivalent to lower bounds on intrinsic curvature tensors in [7]. The following is an exemplary curvature condition implying  $CD(\rho_1, \rho_2, \kappa, d)$  on CR manifold.

**Proposition 2.2.** [7] *Let  $(\mathbb{M}, \theta)$  be a complete CR Sasakian manifold with real dimension  $2n + 1$ . The Tanaka-Webster Ricci tensor satisfies the bound*

$$\text{Ric}_x(v, v) \geq \rho_1 |v|^2, \forall x \in \mathbb{M}, \forall v \in \mathcal{H}_x,$$

*if and only if the curvature dimension inequality  $CD(\rho_1, \frac{d}{4}, 1, d)$  holds with  $d = 2n$  and  $\Gamma^Z(f) = (Tf)^2$  and the hypothesis (H.1), (H.2), (H.3) are satisfied.*

With the curvature inequality condition assumed, various aspects on sub-Riemannian manifolds have been discovered in [7],[4],[5],[6],[8],[9],[10]. In particular, we have the following essential properties - two-sided heat kernel bounds and volume doubling property of balls with exponential term.

**Proposition 2.3.** [6] *If we assume the curvature condition  $CD(-K, \rho_2, \kappa, d)$ ,  $K > 0$  on  $\mathbb{M}$ , for any  $x, y \in \mathbb{M}$ ,  $t > 0$ ,  $r > 0$ ,*

$$p_t(x, y) \geq \frac{C_1}{\mu(B(x, \sqrt{t}))} \exp \left( -\frac{D}{2d} \frac{d(x, y)^2}{t} - C_2 K(t + d(x, y)^2) \right) \quad (2.7)$$

$$p_t(x, y) \leq \frac{C_3}{\mu(B(x, \sqrt{t}))^{1/2} \mu(B(y, \sqrt{t}))^{1/2}} \exp \left( C_4 Kt - \frac{d(x, y)^2}{5t} \right) \quad (2.8)$$

$$\mu(B(x, 2r)) \leq C_{d1} \exp(C_{d2} K r^2) \mu(B(x, r)), \quad (2.9)$$

where  $D = d \left( 1 + \frac{3\kappa}{2\rho_2} \right)$ ,  $C_1, C_2, C_3, C_4, C_{d1}, C_{d2}$  are positive and determined by  $\rho_2, \kappa, d$ .

Denote  $Q = \log_2 C_{d1}$ . (2.9) implies that for any  $\lambda > 1$ ,

$$\begin{aligned} \frac{\mu(B(x, \lambda r))}{\mu(B(x, r))} &\leq C_{d1}^{\lceil \log_2 \lambda \rceil} \exp(C_{d2} K \sum_{i=0}^{\lceil \log_2 \lambda \rceil - 1} (2^i r)^2) \\ &\leq C_{d1} \lambda^Q \exp\left(\frac{4C_{d2}}{3} K (\lambda r)^2\right). \end{aligned} \quad (2.10)$$

This doubling property allows us to estimate  $\mu(B(x, \sqrt{t}))$  by  $\mu(B(y, \sqrt{t}))$ :

$$\begin{aligned} \mu(B(x, \sqrt{t})) &\leq \mu(B(y, \sqrt{t} + d(x, y))) \\ &\leq \mu(B(y, \sqrt{t})) 2C_{d1} \left( 1 + \frac{d(x, y)^2}{t} \right)^{Q/2} \exp \left( \frac{8C_{d2}}{3} K(t + d(x, y)^2) \right). \end{aligned}$$

So we modify the upper bound of heat kernel (2.8) with the volume of a single ball, i.e., for  $C_5, C_6 > 0$  depending on  $\rho_2, \kappa, d$ ,

$$p_t(x, y) \leq \frac{C_5}{\mu(B(x, \sqrt{t}))} \exp \left( C_6 K(t + d(x, y)^2) - \frac{d(x, y)^2}{6t} \right). \quad (2.11)$$

Note that  $1 + A \leq C(\epsilon)e^{\epsilon A}$  for  $\forall \epsilon > 0, A \geq 0$  is applied.

**Remark 2.4.** *As mentioned in [6], the square in the exponent of volume doubling property might not be optimal. For instance, in the Riemannian manifold with Ricci tensor bounded below by  $-K < 0$ , by the Bishop-Gromov comparison theorem we have  $V(x, \lambda r) \leq V(x, r) \lambda^n \exp(\sqrt{(n-1)K}(\lambda r))$  where  $\lambda > 1$  and  $V(x, r)$  is the Riemannian measure of the ball  $B(x, r)$ .*

*This yields the difference of the exponent in (1.1) and (1.2).*

**Remark 2.5.** ([4]) Notice that the positive solution  $u$  carries additional condition in the Li-Yau type inequality, Corollary 1.4. Due to the technical reason in the proof of Theorem 6.1 in [7],  $u$  needs to be contained in  $\mathcal{A}_\epsilon = \{f \in C_b^\infty(\mathbb{M}) : f - \epsilon \geq 0, \sqrt{\Gamma(f - \epsilon)}, \sqrt{\Gamma^Z(f - \epsilon)} \in L^2(\mathbb{M})\}$ . Same restriction is required for log-Sobolev inequality in [4].

### 3 Poincaré inequality on the ball

#### 3.1 Lower bound of the Dirichlet heat kernel on the ball

Throughout this section,  $L$  satisfies  $CD(-K, \rho_2, \kappa, d)$ ,  $K > 0$  on  $\mathbb{M}$ .

To adapt Kusuoka and Stroock's idea [27], the necessary ingredients will be two-sided heat kernel bound (2.7), (2.8) and doubling (2.10).

Denote  $B = B(x_0, r)$ , sub-Riemannian ball centered at  $x_0$  with radius  $r$ . On the ball  $B$ , the Dirichlet heat kernel  $p_t^{B,D}(x, y)$  will be defined by the transition probability

$$p_t^{B,D}(x, y) d\mu(y) = P[\zeta > t, X(t) \in d\mu(y)],$$

where  $X(t)$  is the associated Markov process of the semigroup operator  $P_t = e^{tL}$  with  $X(0) = x$ , and the lifetime of  $X$  in  $B$  is  $\zeta = \inf\{t > 0, X(t) \notin B\}$ .

First, the lower bound of the Dirichlet heat kernel for close  $x, y$  can be obtained by the argument of Kusuoka and Stroock [27]:

**Lemma 3.1.** *For any  $k \in (0, 1)$ , there exists  $C_\alpha = C(k, \rho_2, \kappa, d) \in (1, \infty)$  such that for any  $x_0 \in \mathbb{M}$ ,  $r > 0$  and  $\alpha = \sqrt{\frac{1}{C_\alpha(Kr^2+1)}} \in (0, 1)$ , the Dirichlet heat kernel on  $B = B(x_0, r)$  has lower bound*

$$p_t^{B,D}(x, y) \geq \frac{c}{\mu(B(x, \sqrt{t}))} \exp\left(-C \frac{d(x, y)^2}{t}\right) \quad (3.12)$$

for all  $t \in (0, (\alpha r)^2]$  and  $x, y \in B(x_0, kr)$  such that  $d(x, y) \leq \alpha r$ .

Here  $c, C > 0$  depend only on  $\rho_2, \kappa, d$ .

*Proof.* Let  $\alpha = (C_\alpha(Kr^2 + 1))^{-\frac{1}{2}} \in (0, 1)$  with some  $C_\alpha > 1$  which will be determined later. Note that  $K(\alpha r)^2 \leq C_\alpha^{-1} \leq 1$ .

Let  $d(x, y) \leq \alpha r$  and  $t \leq (\alpha r)^2$ . The Dirichlet heat kernel can be written by the heat kernel of  $\mathbb{M}$  and the lifetime of the process in the domain. That is,

$$p_t^{B,D}(x, y) = p_t(x, y) - \mathbb{E}^x[p_{t-\zeta}(X(\zeta), y), \zeta < t], \quad \zeta = \inf\{t > 0, X(t) \notin B(x_0, r)\}.$$

The lower bound (2.7) on the heat kernel  $p_t(x, y)$  over the whole manifold yields

$$p_t(x, y) \geq \frac{C_1 e^{-2C_2}}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{D}{2d} \frac{d(x, y)^2}{t}\right).$$

If we use upper bound (2.8) on the heat kernel in the expectation, we have

$$p_{t-\zeta}(X(\zeta), y) \leq \frac{C_3 \exp\left(C_4 K(t-\zeta) - \frac{d(X(\zeta), y)^2}{5(t-\zeta)}\right)}{\mu(B(X(\zeta), \sqrt{t-\zeta}))^{1/2} \mu(B(y, \sqrt{t-\zeta}))^{1/2}}.$$

The balls can be replaced by concentric balls using the doubling property (2.10) as follows.

$$\begin{aligned} \frac{\mu(B(x, \sqrt{t}))}{\mu(B(X(\zeta), \sqrt{t-\zeta}))} &\leq \frac{\mu(B(X(\zeta), 3r))}{\mu(B(X(\zeta), \sqrt{t-\zeta}))} \leq C_{d1} \left(\frac{3r}{\sqrt{t-\zeta}}\right)^Q \exp\left(\frac{4C_{d2}K}{3}(3r)^2\right), \\ \frac{\mu(B(x, \sqrt{t}))}{\mu(B(y, \sqrt{t-\zeta}))} &\leq \frac{\mu(B(y, 2\alpha r))}{\mu(B(y, \sqrt{t-\zeta}))} \leq C_{d1} \left(\frac{2\alpha r}{\sqrt{t-\zeta}}\right)^Q \exp\left(\frac{4C_{d2}K}{3}(2\alpha r)^2\right). \end{aligned}$$

With  $d(X(\zeta), y) \geq r(1-k)$  and  $t-\zeta \leq t \leq (\alpha r)^2$ , the above controls imply that

$$\begin{aligned} \mathbb{E}^x[p_{t-\zeta}(X(\zeta), y), \zeta < t] \\ \leq \frac{C_3 C_{d1} e^{3C_{d2}+C_4}}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{r^2(1-k)^2}{10t} + 6C_{d2}Kr^2\right) \cdot C_\nu \exp\left(-\nu \frac{1}{\alpha^2}\right). \end{aligned}$$

The last term came from  $\mathbb{E}^x \left[ \left( \frac{\sqrt{(3r)(2\alpha r)}}{\sqrt{t-\zeta}} \right)^Q \exp\left(-\frac{r^2(1-k)^2}{10(t-\zeta)}\right) \right] \leq C_\nu \exp\left(-\nu \frac{1}{\alpha^2}\right)$ , which holds if we choose  $C_\nu \geq \left(\frac{60Q}{e(1-k)^2}\right)^{Q/2}$ ,  $\nu \leq \frac{(1-k)^2}{20}$ .

Combining these upper and lower estimates with  $t \leq (\alpha r)^2$ ,  $d(x, y) \leq \alpha r$ ,

$$\begin{aligned} p_t^{B,D}(x, y) \\ \geq \frac{C_1 e^{-2C_2}}{\mu(B(x, \sqrt{t}))} e^{\left(-\frac{D}{2d} \frac{d(x, y)^2}{t}\right)} \left[ 1 - C \exp\left(-\frac{r^2}{t} \left(\frac{(1-k)^2}{10} - \frac{D}{2d} \alpha^2\right) + 6C_{d2}Kr^2 - \frac{\nu}{\alpha^2}\right) \right], \end{aligned}$$

where  $C = C_3 C_{d1} C_\nu C_1^{-1} e^{3C_{d2}+C_4+2C_2}$ .

Choose  $\alpha$  small enough for  $\left[ 1 - C \exp\left(-\frac{r^2}{t} \left(\frac{(1-k)^2}{10} - \frac{D}{2d} \alpha^2\right) + 6C_{d2}Kr^2 - \frac{\nu}{\alpha^2}\right) \right] \geq \frac{1}{2}$ .

Then we conclude

$$p_t^{B,D}(x, y) \geq \frac{c}{\mu(B(x, \sqrt{t}))} \exp\left(-C \frac{d(x, y)^2}{t}\right),$$

for all  $t \leq (\alpha r)^2$ ,  $d(x, y) \leq \alpha r$ , where  $c = 2^{-1} C_1 e^{-2C_2}$ ,  $C = \frac{D}{2d} > 0$  depend on  $\rho_2, \kappa, d$ .

For instance, if we pick large  $C_\alpha = C(\rho_2, \kappa, d, k) > 0$  such as

$$C_\alpha \geq \max \left\{ \frac{\frac{D}{2d} + \ln(2C_3 C_{d1} C_\nu C_1^{-1} e^{3C_{d2}+C_4+2C_2}) + 6C_{d2}}{\nu + \frac{(1-k)^2}{10}}, \left( \frac{2d}{D} \frac{(1-k)^2}{10} \right)^{-1} \right\},$$



then our choice of

$$\alpha^2 = \frac{1}{C_\alpha(Kr^2 + 1)}, \quad (3.13)$$

satisfies the estimates above.  $\square$

Next step is the lower bound of Dirichlet heat kernel for any  $x, y$  in the smaller ball which is followed by the chain argument. Note that our lemma holds for any  $r > 0$  with the exponential square of radius, while the classic lemma holds only for  $0 < r \leq 1$ .

**Lemma 3.2.** *For any  $0 < k < 1$  and  $0 < \delta < 1$ , there exists  $0 < c < 1$ ,  $C > 0$  such that for any  $x_0 \in \mathbb{M}$  and  $r > 0$ , the Dirichlet heat kernel on the ball  $B = B(x_0, r)$  has lower bound*

$$p_t^{B,D}(x, y) \geq \frac{c \exp(-CKr^2)}{\mu(B(x_0, kr))}$$

for all  $x, y \in B(x_0, kr)$  and  $\delta r^2 \leq t \leq r^2$ .

*Proof.* Choosing  $\alpha \in (0, 1)$  of (3.13) in the previous lemma, for all  $t \leq (\alpha r)^2$ ,  $x, y \in B(x_0, kr)$ ,  $d(x, y) \leq \alpha r$ ,

$$p_t^{B,D}(x, y) \geq \frac{c}{\mu(B(x, \sqrt{t}))} \exp(-C \frac{d(x, y)^2}{t}).$$

Now let  $x, y$  be any points in  $B(x_0, kr)$  and  $\delta r^2 \leq t \leq r^2$ . Set  $n = \lceil 16\alpha^{-2} \rceil$ , then  $16\alpha^{-2} \leq n \leq 17\alpha^{-2}$ .

We choose  $\{\xi_i\}_{i=0,1,\dots,2n} \subset B(x_0, kr)$  such that

$$\begin{aligned} \xi_0 &= x, & \xi_n &= x_0, & \xi_{2n} &= y, \\ d(\xi_k, \xi_{k+1}) &\leq \frac{r}{n} \leq \frac{\alpha r}{4}. \end{aligned}$$

Let  $\tau = \frac{t}{2n}$ . Since  $\sqrt{\tau} \leq \frac{\alpha r}{4}$ , if  $\eta_k \in B(\xi_k, \sqrt{\tau})$ , then  $d(\eta_k, \eta_{k+1}) \leq \alpha r$ .

By the previous lemma,

$$\begin{aligned} p_\tau^{B,D}(\eta_k, \eta_{k+1}) &\geq \frac{c}{\mu(B(\eta_k, \sqrt{\tau}))} \exp(-C \frac{d(\eta_k, \eta_{k+1})^2}{\tau}) \\ &\geq \frac{cC_{d1}^{-1} \exp(-C_{d2}K(\alpha r)^2)}{\mu(B(\xi_k, \sqrt{\tau}))} \exp(-C \frac{d(\eta_k, \eta_{k+1})^2}{\tau}). \end{aligned}$$

And we see that

$$\frac{d(\eta_k, \eta_{k+1})^2}{\tau} \leq \left( \frac{d(\xi_k, \xi_{k+1})}{\sqrt{\tau}} + 2 \right)^2 \leq \left( \frac{r\sqrt{2}}{\sqrt{nt}} + 2 \right)^2 \leq \frac{4}{\delta n} + 8.$$

Observing

$$p_t^{B,D}(x, y) \geq \int_{B(\xi_{2n-1}, \sqrt{\tau})} \cdots \int_{B(\xi_1, \sqrt{\tau})} p_\tau^{B,D}(x, \eta_1) \cdot p_\tau^{B,D}(\eta_1, \eta_2) \cdots p_\tau^{B,D}(\eta_{2n-1}, y) d\eta_1 \cdots d\eta_{2n-1},$$

we obtain

$$p_t^{B,D}(x, y) \geq \frac{1}{\mu(B(x_0, \sqrt{\tau}))} \left( cC_{d1}^{-1} \exp(-C(\frac{4}{\delta n} + 8) - C_{d2}K(\alpha r)^2) \right)^{2n}.$$

Doubling property (2.10) yields

$$\frac{1}{\mu(B(x_0, \sqrt{\tau}))} \geq \frac{1}{\mu(B(x_0, r))} \geq \frac{C_{d1}^{-1} k^Q \exp(-\frac{4C_{d2}}{3} Kr^2)}{\mu(B(x_0, kr))}.$$

Also since  $cC_{d1}^{-1} \exp(-8C) < 1$  and  $n \leq 17\alpha^{-2} = 17C_\alpha(Kr^2 + 1)$  from (3.13),

$$\begin{aligned} & \left( cC_{d1}^{-1} \exp\left(-C(\frac{4}{\delta n} + 8) - C_{d2}K(\alpha r)^2\right) \right)^{2n} \\ & \geq \exp\left(-\frac{8C}{\delta} - (8C - \ln(cC_{d1}^{-1}) + C_{d2}) \cdot 34C_\alpha(Kr^2 + 1)\right). \end{aligned}$$

This concludes our lemma

$$p_t^{B,D}(x, y) \geq \frac{c' \exp(-C' Kr^2)}{\mu(B(x_0, kr))},$$

where

$$\begin{aligned} c' &= C_{d1}^{-1} k^Q \exp\left(-\frac{8C}{\delta} - (8C - \ln(cC_{d1}^{-1}) + C_{d2})(34C_\alpha)\right), \\ C' &= \frac{4C_{d2}}{3} + (8C - \ln(cC_{d1}^{-1}) + C_{d2})(34C_\alpha) \end{aligned}$$

are determined by  $\rho_2, \kappa, d, k, \delta$ . □

### 3.2 proof of Theorem 1.1

In this section, we utilize Dirichlet, Neumann heat semigroup which can be found in [40],[35],[27],[22], then we will follow the arguments in [27] to prove Poincaré inequality (1.2).

Let  $B = B(x_0, r)$ . Define a subspace  $D^\infty \subset C^\infty(B)$  as a collection of functions  $f$  satisfying  $-\int_B g Lf d\mu = \int_B \Gamma(g, f) d\mu$  for  $\forall g \in C^\infty(B)$ . Note that  $C_0^\infty(B) \subset D^\infty \subset C^\infty(B)$ .

The Dirichlet form  $\mathcal{E}(f, g) = \int_B \Gamma(f, g) d\mu$  on  $D^\infty$  is closable in  $L^2(B)$ , and by closing it we gain a Dirichlet form and associated Markov heat semigroup  $P_t^{B,N}$  with Neumann boundary condition.

If we denote  $p_t^{B,N}$  by the Neumann heat kernel over  $B$ , it will be a smooth kernel of the Neumann heat semigroup and its associated transition probability function. Naturally, since  $C_0^\infty(B) \subset D^\infty$ , the Neumann heat kernel dominates the Dirichlet heat kernel, i.e.,  $p_t^{B,N} \geq p_t^{B,D}$ .

*Proof of Theorem 1.1 .* We will prove the inequality with  $B(x_0, r/2)$  on the left hand side. Then by the Whitney type covering lemma (section 5 in [24]), we can match the balls on the both sides. The Whitney decomposition only requires a doubling property in the domain of argument. In  $B(x_0, 10r)$ , the doubling property holds with fixed constant  $C_{d1} \exp(C_{d2}K(10r)^2)$ , which will be multiplied at the end following the argument.

From the previous lemma, for  $x, y \in B(x_0, r/2)$ ,

$$p_{r^2}^{B(x_0, r), N}(x, y) \geq \frac{ce^{-CKr^2}}{\mu(B(x_0, r/2))}.$$

For any  $f \in C^\infty(B)$  and  $x \in B(x_0, r/2)$ ,

$$\begin{aligned} & P_{r^2}^{B(x_0, r), N} (f - P_{r^2}^{B(x_0, r), N} f(x))^2(x) \\ & \geq \frac{ce^{-CKr^2}}{\mu(B(x_0, r/2))} \int_{B(x_0, r/2)} (f(y) - P_{r^2}^{B(x_0, r), N} f(x))^2 d\mu(y) \\ & \geq \frac{ce^{-CKr^2}}{\mu(B(x_0, r/2))} \int_{B(x_0, r/2)} (f(y) - f_{B(x_0, r/2)})^2 d\mu(y). \end{aligned}$$

On the other hand,

$$\begin{aligned} & \int_{B(x_0, r)} P_{r^2}^{B(x_0, r), N} (f - P_{r^2}^{B(x_0, r), N} f(x))^2(x) d\mu(x) \leq \int_{B(x_0, r)} (f^2 - P_{r^2}^{B(x_0, r), N} f(x)^2) d\mu(x) \\ & = \int_0^{r^2} \int_{B(x_0, r)} -\frac{d}{dt} (P_t^{B(x_0, r), N} f(x))^2 d\mu(x) dt \\ & = \int_0^{r^2} \int_{B(x_0, r)} -2P_t^{B(x_0, r), N} f(x) L P_t^{B(x_0, r), N} f(x) d\mu(x) dt \\ & = \int_0^{r^2} \int_{B(x_0, r)} 2\Gamma(P_t^{B(x_0, r), N} f(x)) d\mu(x) dt \\ & \leq 2r^2 \int_{B(x_0, r)} \Gamma(f) d\mu, \end{aligned}$$

where the last inequality comes from  $\frac{d}{dt} \Gamma(P_t f) \leq 0$ . And we obtain our desired conclusion

$$\int_{B(x_0, r/2)} (f(x) - f_{B(x_0, r/2)})^2 d\mu(x) \leq C'_{p1} r^2 e^{C'_{p2} K r^2} \int_{B(x_0, r)} \Gamma(f) d\mu,$$

with  $C'_{p1} = 2/c$ ,  $C'_{p2} = C$ . □

### 3.3 Sobolev inequality and $L^p$ mean value estimate

As a consequence of Theorem 1.1, this section is dedicated to the Sobolev inequality and  $L^p$  mean value inequality for subharmonic functions, which will be essential to prove Theorem 1.2 and 1.3. Throughout this paper, harmonic (resp. subharmonic) functions are  $f \in \text{Dom}(L)$  satisfying  $Lf = 0$  (resp.  $Lf \geq 0$ ).

Assume that  $L$  satisfies  $CD(-K, \rho_2, \kappa, d)$ ,  $K > 0$  on  $\mathbb{M}$ . We have Poincaré inequality (1.2) and exponential doubling property (2.9).

With these two ingredients, one can derive local Sobolev inequality in [36],[37]. This is a classic path to Moser's iteration for Harnack's inequality. See theorem 2.2 in [36] and section.10 in [37] (also the last section of [42]).

Note that in [19], one can find that the weak  $L^1$  Poincaré inequality with the doubling property derives the isoperimetric inequality (and Sobolev inequalities) in Carnot-Carathéodory spaces.

**Proposition 3.3** ([36],[37], Sobolev inequality on balls). *If the Poincaré inequality (1.2) and the volume doubling condition (2.9) are satisfied for any  $r > 0$ , then for any  $x \in \mathbb{M}$ ,  $0 < r$ ,  $f \in C_0^\infty(B(x, r))$ , denoting  $B = B(x, r)$ ,*

$$\left( \frac{1}{\mu(B)} \int_B |f|^{\frac{2Q}{Q-2}} d\mu \right)^{\frac{Q-2}{2Q}} \leq C_1 r e^{C_e K r^2} \left( \frac{1}{\mu(B)} \int_B (\Gamma(f) + r^{-2} |f|^2) d\mu \right)^{\frac{1}{2}}, \quad (3.14)$$

where  $Q = \log_2 C_{d1}$  in (2.9),  $C_1, C_e > 0$  depend only on  $\rho_2, \kappa, d$ .

Note that this Sobolev inequality can also be obtained by following steps of [37].

With  $CD(-K, \rho_2, \kappa, d)$  assumed, by the upper bound of the heat kernel (2.11)

$$p_t^{B,D}(x, y) \leq p_t(x, y) \leq \frac{C_5}{\mu(B(x, \sqrt{t}))} \exp \left( C_6 K(t + d(x, y)^2) - \frac{d(x, y)^2}{6t} \right),$$

where  $B = B(x_0, r)$ . Since  $0 < t \leq r^2$  and  $d(x, y) \leq 2r$  for  $x, y \in B$ , by (2.10)

$$\mu(B(x_0, r)) \leq \mu(B(x, 2r)) \leq C \left( \frac{r}{\sqrt{t}} \right)^Q \exp(cK r^2) \mu(B(x, \sqrt{t})).$$

Therefore, the Dirichlet heat kernel will be bounded from above by

$$p_t^{B,D}(x, y) \leq \frac{C}{\mu(B(x, \sqrt{t}))} e^{cK r^2} \leq \frac{C'}{\mu(B(x_0, r))} r^Q t^{-Q/2} e^{c'K r^2}.$$

Proposition 10.1 in [37] (also [42]) states that

$$\begin{aligned} \|P_t^{B,D}\|_{1 \rightarrow \infty} &\leq C_0 t^{-Q/2}, \quad \forall 0 < t < t_0 \\ \implies \|f\|_{\frac{2Q}{Q-2}} &\leq C_0^{1/Q} \left( C \|\sqrt{\Gamma(f)}\|_2 + t_0^{-1/2} \|f\|_2 \right), \quad \forall f \in C_0^\infty(B). \end{aligned}$$

Taking  $C_0 = \frac{C'}{\mu(B(x_0, r))} r^Q e^{c' K r^2}$  and  $t_0 = r^2$  with  $A + B \leq (2A^2 + 2B^2)^{1/2}$ , the Sobolev inequality (3.14) is proved.

Once the local Sobolev embedding is acquired, our goal of this section,  $L^p$  mean value estimate, can be obtained through the Moser's iteration. One can find the arguments for the Riemannian case in [35].

In [37],[36], the author obtained parabolic  $L^p$  mean value estimate. But in our context,  $L^p$  mean value estimate for subsolution of  $L$ -Laplace equation will be enough.

**Lemma 3.4** (One step of Moser's iteration). *We assume that  $\mathbb{M}$  satisfies  $CD(-K, \rho_2, \kappa, d)$ ,  $K > 0$ . For any subharmonic function  $u(x) \geq 0$ , i.e.  $Lu(x) \geq 0$ , and  $0 < R_1 < R_2 \leq R$ ,  $p \geq 2$ ,*

$$\int_{B(R_1)} u^{p\theta} d\mu \leq C_2 e^{2C_e K R^2} \frac{R^2}{(R_2 - R_1)^2} V^{1-\theta} \left( \int_{B(R_2)} u^p d\mu \right)^\theta, \quad (3.15)$$

where  $\theta = 1 + \frac{2}{Q}$ ,  $B(\cdot) = B(x_0, \cdot)$ ,  $V = \mu(B(R))$ .

**Remark 3.5.** *For any  $0 < R_1 < R_2 < \infty$ , there exists a Lipschitz continuous cut-off function  $\psi \geq 0$  satisfying  $\psi|_{B(R_1)} = 1$ ,  $\text{supp } \psi \subset B(R_2)$ ,  $\sqrt{\Gamma(\psi)} \leq \frac{C}{R_2 - R_1}$  almost everywhere for some  $C > 0$  which is independent to  $R_1, R_2$ . See theorem 1.5 in [20], lemma 3.6 in [13] and [39].*

*Proof.* Denote  $B = B(R)$ . Let  $\psi \geq 0$  be a cut-off function satisfying  $\psi|_{B(R_1)} = 1$ ,  $\text{supp } \psi \subset B(R_2)$ ,  $\sqrt{\Gamma(\psi)} \leq \frac{C}{R_2 - R_1}$  almost everywhere. Choose a test function  $\phi = \psi^2 u^{p-1} \geq 0$ , and the subharmonicity condition of  $u$  implies

$$0 \leq (Lu, \phi) = \int_B -\Gamma(u, \psi^2 u^{p-1}) d\mu = \int_B \left( -(p-1) \psi^2 u^{p-2} \Gamma(u) - 2\psi u^{p-1} \Gamma(u, \psi) \right) d\mu.$$

Hence, by Cauchy-Schwarz inequality

$$\begin{aligned} \frac{p-1}{2} \int_B \psi^2 u^{p-2} \Gamma(u) d\mu &\leq \int_B \psi u^{p-1} \sqrt{\Gamma(u) \Gamma(\psi)} d\mu \\ &\leq \left( \int_B \psi^2 u^{p-2} \Gamma(u) d\mu \right)^{1/2} \left( \int_B u^p \Gamma(\psi) d\mu \right)^{1/2}. \end{aligned}$$

So, we have

$$\frac{(p-1)^2}{4} \int_B \psi^2 u^{p-2} \Gamma(u) d\mu \leq \int_B u^p \Gamma(\psi) d\mu. \quad (3.16)$$

On the other hand, if we apply Hölder inequality and the local Sobolev inequality (3.14)

on  $\psi u^{p/2}$ , it will give

$$\begin{aligned}
\frac{1}{\mu(B)} \int_B |\psi u^{p/2}|^{2(1+\frac{2}{Q})} d\mu &\leq \left( \frac{1}{\mu(B)} \int_B |\psi u^{p/2}|^{\frac{2Q}{Q-2}} d\mu \right)^{\frac{Q-2}{Q}} \left( \frac{1}{\mu(B)} \int_B |\psi u^{p/2}|^2 d\mu \right)^{\frac{2}{Q}} \\
&\leq C_1^2 R^2 e^{2C_e K R^2} \left( \frac{1}{\mu(B)} \int_B (\Gamma(\psi u^{p/2}) + R^{-2} |\psi u^{p/2}|^2) d\mu \right) \left( \frac{1}{\mu(B)} \int_B \psi^2 u^p d\mu \right)^{\frac{2}{Q}} \\
&\leq C_1^2 R^2 e^{2C_e K R^2} \left( \frac{1}{\mu(B)} \int_B \Gamma(\psi u^{p/2}) d\mu + R^{-2} \frac{1}{\mu(B)} \int_{\text{supp}(\psi)} u^p d\mu \right) \\
&\quad \cdot \|\psi\|_\infty^{\frac{4}{Q}} \left( \frac{1}{\mu(B)} \int_{\text{supp}(\psi)} u^p d\mu \right)^{\frac{2}{Q}}.
\end{aligned}$$

Using (3.16), the gradient term can be written by

$$\begin{aligned}
\int_B \Gamma(\psi u^{p/2}) d\mu &\leq \int_B 2 \left( u^p \Gamma(\psi) + \frac{p^2}{4} \psi^2 u^{p-2} \Gamma(u) \right) d\mu \\
&\leq \left( 2 + \frac{2p^2}{(p-1)^2} \right) \int_B u^p \Gamma(\psi) d\mu \leq \left( 2 + \frac{2p^2}{(p-1)^2} \right) \|\Gamma(\psi)\|_\infty \int_{\text{supp} \psi} u^p d\mu.
\end{aligned}$$

Therefore, given  $\text{supp} \psi \subset B(R_2)$ ,  $\psi|_{B(R_1)} = 1$ ,  $0 \leq \psi \leq 1$  and  $\|\Gamma(\psi)\|_\infty \leq \frac{C^2}{(R_2 - R_1)^2}$ , we obtain

$$\begin{aligned}
&\frac{1}{\mu(B)} \int_{B(R_1)} u^{p\theta} d\mu \\
&\leq C_1^2 e^{2C_e K R^2} \left( \left( 2 + \frac{2p^2}{(p-1)^2} \right) \frac{C^2 R^2}{(R_2 - R_1)^2} + 1 \right) \left( \frac{1}{\mu(B)} \int_{B(R_2)} u^p d\mu \right)^\theta \\
&\leq 11C^2 C_1^2 e^{2C_e K R^2} \frac{R^2}{(R_2 - R_1)^2} \left( \frac{1}{\mu(B)} \int_{B(R_2)} u^p d\mu \right)^\theta,
\end{aligned}$$

where  $\theta = 1 + \frac{2}{Q}$ . Note that we can assume that  $C > 1$  without loss of generality.

The desired inequality (3.15) is proved with  $C_2 = 11C^2 C_1^2$ .  $\square$

Now by iterating the above lemma, we prove  $L^p$  mean value estimate.

**Theorem 3.6** ( $L^p$  mean value inequality,  $p \geq 2$ ). *For any  $0 < \delta < 1$ , any  $p \geq 2$ , and any non-negative subsolution  $u$  of  $Lu = 0$  in a ball  $B(R)$  of volume  $V$ ,*

$$\sup_{\delta B} \{u^p\} \leq C_3 e^{QC_e K R^2} (1 - \delta)^{-Q} \left( V^{-1} \int_B u^p d\mu \right). \quad (3.17)$$

*Proof.* For  $i = 0, 1, 2, \dots$ , set  $p_i = p\theta^i$  where  $\theta = 1 + \frac{2}{Q}$ .

And let  $R_0 = R$ ,  $R_i - R_{i+1} = \frac{(1-\delta)R}{2^{(i+1)}}$ , i.e.

$$R_i = R - \sum_{j=1}^i \frac{(1-\delta)R}{2^j} = R - (1-\delta)R(1 - \frac{1}{2^i}) = \delta R + \frac{(1-\delta)R}{2^i}.$$

By Lemma 3.4,

$$\int_{B(R_{i+1})} u^{p_{i+1}} d\mu \leq \frac{C_2 2^{2(i+1)} V^{1-\theta}}{(1-\delta)^2} e^{2C_e K R^2} \left( \int_{B(R_i)} u^{p_i} d\mu \right)^\theta,$$

This yields

$$\left( \int_{B(R_{i+1})} u^{p_{i+1}} d\mu \right)^{\frac{1}{p_{i+1}}} \leq \left( \frac{C_2 e^{2C_e K R^2} V^{1-\theta}}{(1-\delta)^2} \right)^{\sum_{j=1}^{i+1} \frac{1}{p\theta^j}} 2^{2\sum_{j=1}^{i+1} \frac{j}{p\theta^j}} \left( \int_{B(R)} u^p d\mu \right)^{\frac{1}{p}}.$$

Simple computation shows that

$$\sum_{j=1}^{\infty} \frac{1}{\theta^j} = \frac{1}{\theta-1} = \frac{Q}{2}, \quad \sum_{j=1}^{\infty} \frac{j}{\theta^j} = \frac{\theta}{(\theta-1)^2} = \frac{Q(Q+2)}{4}, \quad \lim_{i \rightarrow \infty} R_i = \delta R$$

$$\lim_{i \rightarrow \infty} \left( \int_{B(R_{i+1})} u^{p_{i+1}} d\mu \right)^{\frac{1}{p_{i+1}}} = \sup_{\delta B} \{u\},$$

Conclusively, where  $C_3 = C_2^{\frac{Q}{2}} 2^{\frac{Q(Q+2)}{2}}$ , we have

$$\sup_{\delta B} \{u\} \leq \left( C_3 e^{Q C_e K R^2} (1-\delta)^{-Q} V^{-1} \right)^{\frac{1}{p}} \left( \int_{B(R)} u^p d\mu \right)^{\frac{1}{p}}.$$

□

**Corollary 3.7** ( $L^p$  mean value inequality,  $0 < p < 2$ ).  $L^p$  mean value inequality (3.17) also holds for any  $0 < p < 2$  with the constant  $C_3$  replaced by some  $C_4 = C(Q, p)$ . In particular, for  $p = 1$

$$\sup_{\delta B} \{u\} \leq C_m e^{c_m K R^2} (1-\delta)^{-Q} \left( \frac{1}{\mu(B)} \int_B u d\mu \right),$$

where  $C_m, c_m > 0$  depend only on  $\rho_2, \kappa, d$ .

*Proof.* Let  $\frac{1}{2} < \sigma < 1$  and  $\rho = \sigma + (1 - \sigma)/4$ .

By Theorem 3.6 (we can pick  $R_0 = \rho R$  in the proof) and  $\int_B u^2 d\mu \leq \sup_B \{u^{2-p}\} \int_B u^p d\mu$ ,

$$\begin{aligned} \sup_{\sigma B} \{u\} &\leq C e^{cKR^2} (1 - \sigma)^{-\frac{Q}{2}} \left( V^{-1} \int_{\rho B} u^2 d\mu \right)^{\frac{1}{2}} \\ &\leq \left( C V^{-\frac{1}{2}} \left( \int_B u^p d\mu \right)^{\frac{1}{2}} e^{cKR^2} \right) (1 - \sigma)^{-\frac{Q}{2}} \sup_{\rho B} \{u^{1-\frac{p}{2}}\}. \end{aligned}$$

Now fix  $\delta \in (\frac{1}{2}, 1)$  and set  $\sigma_0 = \delta$ ,  $\sigma_{i+1} = \sigma_i + (1 - \sigma_i)/4 = \sigma_i + (\frac{3}{4})^i (1 - \delta)/4$ . Then

$$\sup_{\sigma_i B} \{u\} \leq \Lambda \left( \frac{4}{3} \right)^{Qi/2} (1 - \delta)^{-\frac{Q}{2}} \left( \sup_{\sigma_{i+1} B} \{u\} \right)^{1-\frac{p}{2}},$$

where  $\Lambda = \left( C(V^{-1} \int_B u^p d\mu)^{\frac{1}{2}} e^{cKR^2} \right)$ .

Finally, the same iteration of Theorem 3.6 yields

$$\begin{aligned} \sup_{\delta B} \{u\} &\leq \left( \frac{4}{3} \right)^{\frac{2Q}{p^2}} \Lambda^{\frac{2}{p}} (1 - \delta)^{-\frac{Q}{p}} \\ &= \left( \frac{4}{3} \right)^{\frac{2Q}{p^2}} C^{\frac{2}{p}} e^{\frac{2c}{p}KR^2} (1 - \delta)^{-\frac{Q}{p}} \left( V^{-1} \int_B u^p d\mu \right)^{\frac{1}{p}}. \end{aligned}$$

□

## 4 Uniqueness of the positive solution

### 4.1 Minimality of the heat semigroup for positive solutions

To prove Theorem 1.2, we reduce the question to the zero initial data. Following Lemma 4.1 enables the reduction. This section is based on the idea of [14].

**Lemma 4.1** (minimality of the heat semigroup). *Let  $u \in C(\mathbb{M} \times (0, T))$  be a non-negative supersolution of the heat equation (1.3) with initial data  $f \in L^2_{loc}(\mathbb{M})$ ,  $f \geq 0$ .*

*Then  $P_t f(x) = \int_{\mathbb{M}} p_t(x, y) f(y) d\mu(y)$  is a smooth solution of (1.3) satisfying  $P_t f \xrightarrow{L^2_{loc}} f$  as  $t \rightarrow 0$  and  $u(\cdot, t) \geq P_t f$ .*

*Proof.* For any  $\Omega \Subset \mathbb{M}$ , we denote  $P_t^{\Omega, D}$  the Dirichlet heat semigroup associated with  $L$  on  $\Omega$ . Using the maximum principle for  $P_t^{\Omega, D} f - u(\cdot, t)$ , we have

$$u(x, t) \geq P_t^{\Omega, D} f(x), \quad \forall x \in \Omega$$

Denote  $f_k = f 1_{\Omega_k} \in L^2(\mathbb{M})$  for the exhaustion  $\{\Omega_k\}$ . As shown above,  $u(\cdot, t) \geq P_t^{\Omega_k, D} f \geq P_t^{\Omega_k, D} f_i$  for all  $i$ . Since  $P_t^{\Omega_k, D} f_i \xrightarrow{L^2(\mathbb{M})} P_t f_i$  as  $k \rightarrow \infty$ , we have  $u(\cdot, t) \geq P_t f_i$  almost everywhere for all  $i$ . Therefore,

$$u(\cdot, t) \geq P_t f \quad \text{almost everywhere.}$$



To prove that the smooth  $P_t f$  solves the heat equation, first we see that  $P_t f \in L^1_{loc}(\mathbb{M})$  from the above estimate. Denote

$$u_k = P_t(\min(f, k)1_{\Omega_k}),$$

then  $u_k$  is a smooth solution of the subelliptic heat equation and  $u_k \nearrow P_t f$  as  $k \rightarrow \infty$  at any  $(x, t) \in \mathbb{M} \times (0, T)$ .

For any  $\varphi \in C_0^\infty(\mathbb{M} \times (0, T))$ , since  $P_t f \in L^1_{loc}(\mathbb{M})$ ,

$$|(\partial_t \varphi + L\varphi)u_k| \leq (\sup |\partial_t \varphi + L\varphi|)1_{\text{supp } \varphi} P_t f \in L^1(\mathbb{M}), \quad \forall k \in \mathbb{N}.$$

This allows us to take the limit of the integrand on the left hand side of

$$\int_0^T \int_{\mathbb{M}} (\partial_t \varphi + L\varphi)u_k d\mu dt = \int_0^T \int_{\mathbb{M}} \varphi (L - \partial_t) u_k d\mu dt = 0.$$

Therefore  $P_t f$  is a distributional solution of the subelliptic heat equation, and also it is smooth by the smooth convergence of  $u_k$  to  $P_t f$  and the hypoellipticity of  $L - \partial_t$ .

Once the smoothness of  $P_t f$  and  $u \geq P_t f$  are proved, the initial condition is straightforward as follows : On any  $\Omega \Subset \mathbb{M}$ ,

$$P_t(f1_\Omega) \leq P_t f \leq u(\cdot, t).$$

When  $t \rightarrow 0$ ,  $u \xrightarrow{L^2(\Omega)} f$  and  $P_t(f1_\Omega) \xrightarrow{L^2(\mathbb{M})} f1_\Omega$ . Hence  $P_t f \xrightarrow{L^2(\Omega)} f$ .  $\square$

## 4.2 proof of Theorem 1.2

From the minimality Lemma 4.1, for any non-negative continuous solution  $u$  of (1.3),

$$w(x, t) = u(x, t) - P_t u(x, 0)$$

is a non-negative solution of (1.3) with zero initial data. Thus we can reduce the uniqueness of the positive solution to the zero initial data case.

Let  $w(x, t)$  be any non-negative solution of the heat equation (1.3) with initial data  $f \equiv 0$ . Define  $v(x, t) = \int_0^t w(x, s) ds$ . Our goal is to show  $v \equiv 0$ , and so is  $w$ .

**Remark 4.2.**  $v(x, t) = \int_0^t w(x, s) ds$  is a non-negative solution of the heat equation (1.3) with zero initial data, and subharmonic in  $x$ , i.e.  $Lv(\cdot, t) = \int_0^t Lw(\cdot, s) ds = w(\cdot, t) \geq 0$ .

The following growth estimate condition is originally suggested by Tikhonov for the uniqueness of the solution for the heat equation.

**Proposition 4.3** (Growth estimate of the solution, Tikhonov's condition). *For any  $\epsilon > 0$  and  $0 \leq t \leq \epsilon$ , if  $v \in C(\mathbb{M} \times (0, \epsilon))$  is a non-negative solution of the subelliptic heat equation (1.3) satisfying  $Lv(\cdot, t) \geq 0$ , then*

$$v(x, t) \leq C_1 \exp(C_2 d^2(p, x)),$$

where  $C_1 = C_1(\epsilon) > 0$ ,  $C_2 = C_2(\epsilon) > 0$ , and  $d(p, \cdot)$  is the distance from a fixed  $p \in \mathbb{M}$ .

*Proof.* Let  $B = B(x, d(p, x) + 1)$ . Fix  $T > 0$ . From the minimality Lemma 4.1,

$$v(p, t + T) \geq P_T v(\cdot, t) = \int_M p_T(p, y) v(y, t) d\mu(y) \geq \int_B p_T(p, y) v(y, t) d\mu(y).$$

From the curvature condition  $CD(-K, \rho_2, \kappa, d)$ , the lower bound of heat kernel (2.7) is

$$p_T(p, y) \geq C_3 \exp(-C_4 d^2(p, y)),$$

where  $C_3 = C_3(p, T, K, \rho_2, \kappa, d) > 0$ ,  $C_4 = C_4(T, K, \rho_2, \kappa, d) > 0$ .

By the triangle inequality  $d(p, y) \leq 2d(p, x) + 1$  for  $y \in B$ ,

$$\int_B v(y, t) d\mu(y) \leq C_5 \exp(C_6 d^2(p, x)) v(p, t + T).$$

By  $L^1$  mean value estimate of Corollary 3.7 for the subharmonic function  $v(\cdot, t)$

$$v(x, t) \leq C_7 \exp(C_8 d(p, x)^2) \int_B v(y, t) d\mu(y),$$

where  $C_7, C_8 > 0$  depend on  $K, \rho_2, \kappa, d$ . Therefore, we obtain

$$v(x, t) \leq C_9 \exp(C_{10} d^2(p, x)) v(p, t + T),$$

where the constants depend on  $p, T, K, \rho_2, \kappa, d$ . As  $t$  varies from 0 to  $\epsilon$ ,  $v(p, t + T)$  remains uniformly bounded in  $t$ . So we have the desired conclusion.  $\square$

Together with the previous proposition, the proof of Theorem 1.2 is finished by the following proposition.

**Proposition 4.4.** *If  $v(x, t)$  is a solution of (1.3) with initial  $f(x) \equiv 0$  satisfying*

$$|v(x, t)| \leq C_1 \exp C_2 d^2(p, x)$$

*for some positive  $C_1, C_2$ , then  $v \equiv 0$ .*

Existence of Lipschitz cut-off function and integration by part allow us to follow exactly the same proof of corollary 11.10 in [21].

## 5 Uniqueness of $L^p$ solution

### 5.1 proof of Theorem 1.3, $p > 1$

For  $p = \infty$ , the uniqueness of the  $L^\infty$  solution, or equivalently the stochastic completeness of  $\mathbb{M}$ , can be found in [7]. If  $p \in (1, \infty)$ , without any curvature assumption, the uniqueness follows immediately by adapting the idea of [30].

**Theorem 5.1.** *Let  $v(x, t)$  is a non-negative function defined on  $\mathbb{M} \times (0, T)$  with*

$$\begin{aligned} \left( \frac{\partial}{\partial t} - L \right) v(x, t) &\leq 0 \\ v &\xrightarrow{L^p_{loc}} 0 \text{ as } t \rightarrow 0 \\ v(\cdot, t) &\in L^p(\mathbb{M}) \quad \forall t \in (0, T), \end{aligned}$$

*then  $v(x, t) \equiv 0$  on  $\mathbb{M} \times (0, T)$ .*

*In particular, any  $L^p$  solution of the heat equation is uniquely determined by its initial data in  $L^p(\mathbb{M})$ .*

*Proof.* Fix  $x_0 \in \mathbb{M}$  an arbitrary base point.

From remark 3.5, we choose  $\psi(x) \in C_0(B(x_0, 2R))$  a cut-off function satisfying  $\psi|_{B(x_0, R)} \equiv 1$ ,  $0 \leq \psi \leq 1$ ,  $\|\sqrt{\Gamma(\psi)}\|_\infty \leq \frac{C}{R}$  for some  $C > 0$ .

Since  $v$  is a subsolution with the zero initial data, for any  $\tau \in (0, T)$ ,

$$\begin{aligned} \int_0^\tau \int_{\mathbb{M}} \psi^2(x) v^{p-1}(x, t) L v(x, t) d\mu(x) dt &\geq \int_0^\tau \int_{\mathbb{M}} \psi^2(x) v^{p-1} \frac{\partial v}{\partial t} d\mu(x) dt \\ &= \frac{1}{p} \int_0^\tau \frac{\partial}{\partial t} \left( \int_{\mathbb{M}} \psi^2(x) v^p d\mu(x) \right) dt = \frac{1}{p} \int_{\mathbb{M}} \psi^2(x) v^p(x, \tau) d\mu(x). \end{aligned}$$

On the other hand, integrating by parts yields

$$\begin{aligned} \int_0^\tau \int_{\mathbb{M}} \psi^2(x) v^{p-1}(x, t) L v(x, t) d\mu(x) dt \\ = - \int_0^\tau \int_{\mathbb{M}} 2\psi v^{p-1} \Gamma(\psi, v) d\mu dt - \int_0^\tau \int_{\mathbb{M}} \psi^2 (p-1) v^{p-2} \Gamma(v) d\mu dt. \end{aligned}$$

On the right hand side, observing

$$0 \leq \left( \sqrt{\frac{2}{p-1}} \Gamma(\psi) v - \sqrt{\frac{p-1}{2}} \Gamma(v) \psi \right)^2 \leq \frac{2}{p-1} \Gamma(\psi) v^2 + 2\Gamma(\psi, v) \psi v + \frac{p-1}{2} \Gamma(v) \psi^2,$$

we obtain the following estimate.

$$\begin{aligned} \int_0^\tau \int_{\mathbb{M}} \psi^2(x) v^{p-1}(x, t) L v(x, t) d\mu(x) dt \\ \leq \int_0^\tau \int_{\mathbb{M}} \frac{2}{p-1} \Gamma(\psi) v^p d\mu dt - \int_0^\tau \int_{\mathbb{M}} \frac{p-1}{2} \psi^2 v^{p-2} \Gamma(v) d\mu dt \\ = \int_0^\tau \int_{\mathbb{M}} \frac{2}{p-1} \Gamma(\psi) v^p d\mu dt - \frac{2(p-1)}{p^2} \int_0^\tau \int_{\mathbb{M}} \psi^2 \Gamma(v^{p/2}) d\mu dt. \end{aligned}$$

Combining with the previous conclusion and the assumption  $|\sqrt{\Gamma(\psi)}| \leq \frac{C}{R}$ ,

$$\int_{\mathbb{M}} \psi^2(x) v^p(x, \tau) d\mu(x) + \frac{2(p-1)}{p} \int_0^\tau \int_{\mathbb{M}} \psi^2 \Gamma(v^{p/2}) d\mu dt \leq \frac{2pC^2}{(p-1)R^2} \int_0^\tau \int_{\mathbb{M}} v^p d\mu dt.$$

As  $R \rightarrow \infty$ , since  $\Gamma(v^{p/2}) \geq 0$ , we have

$$\int_{\mathbb{M}} v^p(x, \tau) d\mu(x) = 0 \quad \forall \tau \in (0, T).$$

Thus,  $v \equiv 0$ . □

## 5.2 Hamilton's inequality

Before we move on to  $L^1$  solutions, we will prove the gradient estimate of the logarithm of the heat kernel. We will apply subelliptic version of Hamilton's inequality which was originally proved for closed Riemannian manifolds in [23], then for non-compact Riemannian manifolds in [26].

**Proposition 5.2** (Hamilton's inequality). *Assume that  $\mathbb{M}$  satisfies the curvature condition  $CD(-K, \rho_2, \kappa, d)$ . If a positive solution  $u \in \mathcal{A}_\epsilon$  to the subelliptic heat equation satisfies  $u \leq M$  on  $\mathbb{M} \times (0, T)$  for some  $M > 0$  and  $0 < T \leq \infty$ , one has*

$$t\Gamma(\ln u(x, t)) \leq \left(1 + \frac{2\kappa}{\rho_2} + 2Kt\right) \ln \left(\frac{M}{u(x, t)}\right) \quad (5.18)$$

for all  $(x, t) \in \mathbb{M} \times (0, T)$ .

*Proof.* By Theorem 1.2, it suffices to show that the estimate holds for  $u = P_t f \in \mathcal{A}_\epsilon > 0$ . (See Remark 2.5 for  $\mathcal{A}_\epsilon$ .) We apply the reverse log-Sobolev inequality in [4], i.e.,

$$tP_t f \Gamma(\ln P_t f) + \rho_2 t^2 P_t f \Gamma^Z(\ln P_t f) \leq \left(1 + \frac{2\kappa}{\rho_2} + 2Kt\right) (P_t(f \ln f) - (P_t f) \ln P_t f).$$

Then our desired inequality is instantly deduced by  $P_t(f \ln f) \leq (P_t f) \ln M$  and  $\rho_2 t^2 P_t f \Gamma^Z \geq 0$ . □

**Lemma 5.3.** *If  $\mathbb{M}$  satisfies  $CD(-K, \rho_2, \kappa, d)$ , there exists  $C_h = C_h(\rho_2, \kappa, d) > 0$ ,  $t > 0$ ,  $x, y \in \mathbb{M}$ ,*

$$\Gamma_x(\ln p_t(x, y)) \leq \frac{C_h}{t} \left(1 + \frac{2\kappa}{\rho_2} + Kt\right) \left(K(t + d(x, y)^2) + \frac{d(x, y)^2}{t}\right).$$

*Proof.* Let  $t > 0$  and  $y \in \mathbb{M}$ . Let  $u(x, s) := p_{\frac{t}{2}+s}(x, y)$ , then  $u$  is a smooth, positive solution to the heat equation. From the heat kernel upper bound (2.11), for  $t_0 = \frac{1}{6C_6K}$ ,  $0 < t < t_0$ ,  $0 \leq s \leq \frac{t}{2}$ ,  $\forall x \in \mathbb{M}$ ,

$$\begin{aligned} u(x, s) &\leq \frac{C_5}{\mu(B(y, \sqrt{\frac{t}{2} + s}))} \exp \left( C_6 K \left( \frac{t}{2} + s + d(x, y)^2 \right) - \frac{d(x, y)^2}{6(\frac{t}{2} + s)} \right) \\ &\leq \frac{C_6 e^{1/6}}{\mu(B(y, \sqrt{\frac{t}{2} + s}))} \leq \frac{C'_6}{\mu(B(y, \sqrt{\frac{t}{2}}))} = M. \end{aligned}$$

Moreover  $u(x, s) \leq M$  for all  $s > 0$ , since  $\|P_t\|_{\infty \rightarrow \infty} \leq 1$  for any  $t > 0$ .

By the Hamilton's inequality (5.18), the heat kernel lower bound (2.7) for  $u(x, s)$  with  $s = \frac{t}{2}$  and the doubling property (2.10),

$$\begin{aligned} \frac{t}{2} \Gamma(\ln u(x, \frac{t}{2})) &\leq \left(1 + \frac{2\kappa}{\rho_2} + Kt\right) \ln \left(\frac{M}{u(x, \frac{t}{2})}\right) \\ &\leq \left(1 + \frac{2\kappa}{\rho_2} + Kt\right) \frac{C_h}{2} \left(K(t + d(x, y)^2) + \frac{d(x, y)^2}{t}\right), \end{aligned}$$

where  $C_h = 2 \ln(C'_6 C_1^{-1} C_{d1} 2^{Q/2}) \max(\frac{D}{2d}, \frac{4C_{d2}}{3} + C_2)$ .  $\square$

If we combine the previous lemma with (2.11), we obtain the following simpler statement for small  $t$ , which will be useful in the next section.

**Lemma 5.4.** *Assume  $CD(-K, \rho_2, \kappa, d)$ . For any  $R > 0$ ,  $\beta > 0$  and  $x_0 \in \mathbb{M}$ , there exists  $C > 0$ ,  $t_0 > 0$  such that for  $d(x, y) \geq R/4$ ,  $0 < t < t_0$ ,*

$$\sqrt{\Gamma_y(p_t(x, y))} \leq \frac{C e^{-\beta R^2}}{\mu(B(x, \sqrt{t}))}.$$

### 5.3 proof of Theorem 1.3, $p = 1$

Prior to the uniqueness of  $L^1$  solution for the heat equation, we prove the uniqueness of  $L^1$  harmonic function. Basic idea of the proof comes from [30].

**Remark 5.5.** *We assume the fixed curvature bound  $\rho_1 = -K$  instead of the negative quadratic lower bound of Ricci curvature of [30].*

**Theorem 5.6.** *If  $\mathbb{M}$  satisfies  $CD(-K, \rho_2, \kappa, d)$ , then any  $L^1$  non-negative subharmonic function on  $\mathbb{M}$  must be identically constant.*

*In particular, any  $L^1$  harmonic function on  $\mathbb{M}$  must be identically constant.*

*Proof.* Let  $g \in L^1(\mathbb{M})$  be a non-negative function satisfying  $Lg \geq 0$ , i.e. subharmonic. For any  $t > 0$ ,

$$\begin{aligned} LP_t g(x) &= \int_{\mathbb{M}} (L_x p_t(x, y)) g(y) d\mu(y) \\ &= \int_{\mathbb{M}} \left(\frac{\partial}{\partial t} p_t(x, y)\right) g(y) d\mu(y) = \int_{\mathbb{M}} (L_y p_t(x, y)) g(y) d\mu(y). \end{aligned}$$

We claim the following integration by parts.

$$\int_{\mathbb{M}} (L_y p_t(x, y)) g(y) d\mu(y) = \int_{\mathbb{M}} p_t(x, y) Lg(y) d\mu(y). \quad (5.19)$$

To justify the claim, we observe the following

$$\begin{aligned}
& \left| \int_{\mathbb{M}} \psi_R(y) [g(y)L_y p_t(x, y) - p_t(x, y)Lg(y)] d\mu(y) \right| \\
&= \left| \int_{\mathbb{M}} -[\Gamma(\psi_R g, p_t(x, \cdot)) - \Gamma(\psi_R p_t(x, \cdot), g)] d\mu \right| \\
&= \left| \int_{\mathbb{M}} -[g\Gamma(\psi_R, p_t(x, \cdot)) - p_t(x, \cdot)\Gamma(\psi_R, g)] d\mu \right| \\
&\leq \int_{B(x_0, R+1) \setminus B(x_0, R)} C \left( g\sqrt{\Gamma(p_t(x, \cdot))} + p_t(x, \cdot)\sqrt{\Gamma(g)} \right) d\mu, \tag{5.20}
\end{aligned}$$

where  $\psi_R \geq 0$  is a Lipschitz continuous cut-off function satisfying  $\psi_R|_{B(x_0, R)} = 1$ ,  $\text{supp } \psi_R \subset B(x_0, R+1)$  and  $\sqrt{\Gamma(\psi_R)} \leq C$  almost everywhere for some  $C > 0$  which is independent to  $R > 0$ . (See Remark 3.5.)

It suffices to show that both integrals on the right-hand side vanish as  $R \rightarrow \infty$ . We can consider  $R$  large enough so that  $x \in B(x_0, R/4)$ .

Let  $\varphi$  be a cut-off function for an annulus satisfying  $\varphi|_{B(x_0, R+1) \setminus B(x_0, R)} = 1$ ,  $\varphi|_{B(x_0, R-1) \cup (\mathbb{M} \setminus B(x_0, R+2))} = 0$  and  $\sqrt{\Gamma(\varphi)} \leq C$  almost everywhere. By the subharmonicity of  $g$ ,

$$\begin{aligned}
0 &\leq \int_{\mathbb{M}} \varphi^2 g L g d\mu = -2 \int_{\mathbb{M}} \varphi g \Gamma(\varphi, g) d\mu - \int_{\mathbb{M}} \varphi^2 \Gamma(g) d\mu \\
&\leq \int_{\mathbb{M}} \left[ -\frac{1}{2} (2g\sqrt{\Gamma(\varphi)} - \varphi\sqrt{\Gamma(g)})^2 + 2\Gamma(\varphi)g^2 - \frac{1}{2}\varphi^2\Gamma(g) \right] d\mu \\
&\leq 2 \int_{\mathbb{M}} \Gamma(\varphi)g^2 d\mu - \frac{1}{2} \int_{\mathbb{M}} \varphi^2 \Gamma(g) d\mu.
\end{aligned}$$

Therefore, applying  $L^1$  mean value estimate Corollary 3.7 to  $g$ ,

$$\begin{aligned}
\int_{B(x_0, R+1) \setminus B(x_0, R)} \Gamma(g) d\mu &\leq 4 \int_{\mathbb{M}} \Gamma(\varphi)g^2 d\mu \leq 4C^2 \int_{B(x_0, R+2)} g^2 d\mu \\
&\leq 4C^2 \|g\|_{L^1} \sup_{B(x_0, R+2)} g(y) \leq C' e^{c'KR^2} \frac{1}{\mu(B(x_0, 2R+4))} \|g\|_{L^1}^2.
\end{aligned}$$

By Schwarz inequality,

$$\begin{aligned}
\int_{B(x_0, R+1) \setminus B(x_0, R)} \sqrt{\Gamma(g)} d\mu &\leq \left( \int_{B(x_0, R+1) \setminus B(x_0, R)} \Gamma(g) d\mu \right)^{\frac{1}{2}} (\mu(B(x_0, 2R+4)))^{\frac{1}{2}} \\
&= C e^{cKR^2} \|g\|_{L^1}.
\end{aligned}$$

In addition to this estimate, to bound the second integration in (5.20) we consider the upper bound of heat kernel (2.11):

$$p_t(x, y) \leq \frac{C_5}{\mu(B(x, \sqrt{t}))} \exp \left( C_6 K(t + d(x, y)^2) - \frac{d(x, y)^2}{6t} \right).$$

Combining the above two inequalities, we estimate the second term of (5.20) for small  $0 < t < T = T(K, \rho_2, \kappa, d)$ .

$$\begin{aligned} & \int_{B(x_0, R+1) \setminus B(x_0, R)} p_t(x, \cdot) \sqrt{\Gamma(g)} d\mu \\ & \leq \left( \sup_{y \in B(x_0, R+1) \setminus B(x_0, R)} p_t(x, y) \right) \int_{B(x_0, R+1) \setminus B(x_0, R)} \sqrt{\Gamma(g)} d\mu \\ & \leq \frac{C}{\mu(B(x, \sqrt{t}))} \exp\left(-\alpha \frac{R^2}{t}\right) \|g\|_{L^1} \xrightarrow{R \rightarrow \infty} 0, \end{aligned}$$

with  $d(x_0, x) \leq \frac{R}{4}$ ,  $\frac{R}{2} \leq d(x, y) \leq 4R$  and  $\alpha > 0$ .

For the first term of (5.20),  $L^1$  mean value estimate for  $g$  yields

$$\begin{aligned} & \int_{B(x_0, R+1) \setminus B(x_0, R)} g \sqrt{\Gamma(p_t(x, \cdot))} d\mu \\ & \leq \left( \sup_{B(x_0, R+1) \setminus B(x_0, R)} g \right) \int_{B(x_0, R+1) \setminus B(x_0, R)} \sqrt{\Gamma(p_t(x, \cdot))} d\mu \\ & \leq \left( \frac{C e^{cKR^2}}{\mu(B(x_0, 2R+2))} \|g\|_{L^1} \right) \int_{B(x_0, R+1) \setminus B(x_0, R)} \sqrt{\Gamma(p_t(x, \cdot))} d\mu. \end{aligned}$$

If we apply Lemma 5.4 for  $\beta > cK$ ,

$$\leq \left( \frac{\mu(B(x_0, R+1))}{\mu(B(x_0, 2R+2))} \|g\|_{L^1} \right) \frac{C e^{-\beta' R^2}}{\mu(B(x, \sqrt{t}))} \leq \|g\|_{L^1} \frac{C e^{-\beta' R^2}}{\mu(B(x, \sqrt{t}))} \xrightarrow{R \rightarrow \infty} 0,$$

where  $0 < t < T = T(K, \rho_2, \kappa, d)$  small enough and  $\beta' > 0$ .

Therefore, as  $R \rightarrow \infty$ , the integration of (5.20) vanishes as we desired, and we proved our claim (5.19).

Now since the integration by part (5.19) holds for small  $t$ , we have

$$\frac{\partial}{\partial t} P_t g = L P_t g = P_t (Lg) \geq 0.$$

And by the semigroup property,  $P_t g(x) \geq g(x)$  for all  $t > 0$ ,  $x \in \mathbb{M}$ .

On the other hand, by the stochastic completeness ([7]) of  $P_t$ ,  $\|P_t g\|_{L^1} = \|g\|_{L^1}$ . Therefore  $P_t g = g$ , i.e.  $g$  is harmonic.

For any constant  $\gamma > 0$ ,  $(g - \gamma)_+ = \max(0, g - \gamma) \leq g$  is also a non-negative  $L^1$  subharmonic function. And by the same argument, it is harmonic.  $\min(g, \gamma) = g - (g - \gamma)_+$  is also

non-negative  $L^1$  harmonic function. Observe that  $\min(g, \gamma) \in C^\infty(\mathbb{M})$  for any  $\gamma > 0$  by the hypoellipticity of  $L$ . This is not possible unless  $g$  is constant.

Finally, any harmonic function  $u \in L^1(\mathbb{M})$  is identically constant since  $|u|$  is non-negative  $L^1$  subharmonic function which must be constant by the above.  $\square$

With the uniqueness of  $L^1$  harmonic function, we are ready to prove  $L^1$  uniqueness of the solution for the subelliptic heat equation.

**Theorem 5.7.** *Let  $\mathbb{M}$  satisfy  $CD(-K, \rho_2, \kappa, d)$ . Let  $v : \mathbb{M} \times [0, \infty) \rightarrow \mathbb{R}$  be a non-negative function satisfying*

$$\begin{aligned} \left(L - \frac{\partial}{\partial t}\right) v(x, t) &\geq 0, \quad \|v(\cdot, t)\|_{L^1(\mathbb{M})} < \infty, \quad \forall t > 0, \\ \|v(\cdot, t)\|_{L^1(\mathbb{M})} &\xrightarrow{t \rightarrow 0} 0, \end{aligned}$$

then  $v(x, t) \equiv 0$  on  $\mathbb{M} \times (0, \infty)$ .

*Proof.* For any  $\epsilon > 0$ , denote

$$\psi_\epsilon(x, t) = \max(0, v(x, t + \epsilon) - P_t(v(\cdot, \epsilon))).$$

Then it follows that  $\psi_\epsilon \geq 0$ ,  $\lim_{t \rightarrow 0} \psi_\epsilon(x, t) = 0$ ,  $(L - \frac{\partial}{\partial t}) \psi_\epsilon \geq 0$ .

Fix  $T > 0$ . Define

$$f(x) = \int_0^T \psi_\epsilon(x, t) dt,$$

which satisfies  $Lf(x) = \psi_\epsilon(x, T) - \psi_\epsilon(x, 0) \geq 0$ .

The assumption  $v(\cdot, t) \in L^1(\mathbb{M})$  yields  $\int_0^T \int_{\mathbb{M}} |v(x, t + \epsilon)| d\mu(x) dt < \infty$ . Together with  $\int_0^T \int_{\mathbb{M}} P_t v(x, \epsilon) d\mu dt \leq T \int_{\mathbb{M}} v(x, \epsilon) d\mu(x) < \infty$ , we obtain  $\|f\|_{L^1(\mathbb{M})} < \infty$ .

Now  $f$  is non-negative  $L^1$  subharmonic function, so that we can apply Theorem 5.6 to  $f$  and conclude  $f$  is identically constant. This implies  $0 = Lf(\cdot) = \psi_\epsilon(\cdot, T)$  for arbitrary  $T > 0$ . Hence for any  $t > 0$ ,

$$\begin{aligned} v(x, t + \epsilon) &\leq P_t(v(\cdot, \epsilon))(x) \\ &\leq \|p_t(x, \cdot)\|_\infty \|v(\cdot, \epsilon)\|_{L^1} \leq M \|v(\cdot, \epsilon)\|_{L^1} \xrightarrow{\epsilon \rightarrow 0} 0, \end{aligned}$$

where the uniform bound for  $\|p_t(x, \cdot)\|_\infty$  is found in Lemma 5.3.

Therefore non-negative  $v(x, t)$  must be zero for all  $(x, t) \in \mathbb{M} \times (0, \infty)$ .  $\square$

*Proof of Theorem 1.3,  $p = 1$ .* For any  $L^1$  solution  $u$  of  $(L - \frac{\partial}{\partial t}) u = 0$  with the initial condition  $u \xrightarrow{L^1} f \in L^1(\mathbb{M})$  as  $t \rightarrow 0$ ,

$$v(x, t) := |u(x, t) - P_t f(x)|$$

will be a non-negative  $L^1$  subsolution of the heat equation with  $v \xrightarrow{L^1} 0$  as  $t \rightarrow 0$ .

By the previous theorem,  $v \equiv 0$  on  $\mathbb{M} \times (0, \infty)$ . Therefore,  $u$  is uniquely determined to be  $P_t f$ .  $\square$



**Remark 5.8.** *In [2],[1], the measure contractive definition of Ricci tensor bound and volume comparison theorem (which is not yet established in our framework) are introduced in three dimensional sub-Riemannian spaces. This measure contraction property is extended to higher dimensions in [28].*

*One can find the uniqueness theorem of the positive solution in symmetric local Dirichlet spaces, provided a local parabolic Harnack inequality in [16].*

**Acknowledgements.** *The author is grateful to his advisor, Fabrice Baudoin, for suggesting the problem and for his advice, numerous guidances throughout this work.*

## References

- [1] P. Agrachev, P. Lee: Bishop and Laplacian Comparison Theorems on Three Dimensional Contact Subriemannian Manifolds with Symmetry, arXiv:1105.2206, preprint (2011)
- [2] P. Agrachev, P. Lee: Generalized Ricci curvature bounds on three-dimensional contact sub-Riemannian manifolds, arXiv:0903.2550, preprint (2009)
- [3] D. Bakry, M. Émery: Diffusions hypercontractives, Séminaire de probabilités, XIX, 1983/84, 177-206, Lecture Notes in Math., 1123, Springer, Berlin, (1985)
- [4] F. Baudoin, M. Bonnefont: Log-Sobolev inequalities for subelliptic operators satisfying a generalized curvature dimension inequality, Journ. Func. Anal. 262, no.6, 2646-2676 (2012)
- [5] F. Baudoin, M. Bonnefont, N. Garofalo: A sub-Riemannian curvature-dimension inequality, volume doubling property and the Poincaré inequality, arXiv:1007.1600, to appear in Math. Ann. (2013)
- [6] F. Baudoin, M. Bonnefont, N. Garofalo, I. Munive: Volume and distance comparison theorems for sub-Riemannian manifolds, arXiv:1211.0221, submitted (2012)
- [7] F. Baudoin, N. Garofalo: Curvature-dimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries, arXiv:1101.3590, submitted (2011)
- [8] F. Baudoin, N. Garofalo: A Note on the Boundedness of Riesz Transform for Some Subelliptic Operators, International Mathematics Research Notices, rnn271, 24 pages, doi:10.1093/imrn/rnn271, (2012)
- [9] F. Baudoin, J. Wang: Curvature dimension inequalities and subelliptic heat kernel gradient bounds on contact manifolds, to appear in Potential Analysis, arXiv:1211.3778 (2013)

- [10] F. Baudoin, B. Kim: Sobolev, Poincaré and isoperimetric inequalities for subelliptic diffusion operators satisfying a generalized curvature dimension inequality, to appear in *Revista Matemática Iberoamericana*, arXiv:1203.3789, (2012)
- [11] J.M. Bony: Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés, *Ann. Inst. Fourier* 19(1), 277-304 (1969)
- [12] P. Buser, A note on the isoperimetric constant, *Ann. Sci. Ecole Norm. Sup.* (4) 15, no. 2, 213-230, (1982)
- [13] G. Citti, N. Garofalo, and E. Lanconelli: Harnack's inequality for sum of squares of vector fields plus a potential, *Amer. J. Math.* 115, no. 3, 699-734 (1993)
- [14] H. Donnelly: Uniqueness of positive solutions of the heat equation, *Proc. Amer. Math. Soc.* 99, no. 2, 353-356 (1987)
- [15] S. Dragomir, G. Tomassini: Differential geometry and analysis on CR manifolds, *Progress in Mathematics*, 246. Birkhäuser Boston, Inc., Boston, MA (2006)
- [16] N. Eldredge, L. Saloff-Coste: Widder's representation theorem for symmetric local Dirichlet spaces, arXiv:1204.1926, to appear in *J. Theoret. Probab.* (2012)
- [17] C. Fefferman, D.H. Phong: Subelliptic eigenvalue problems, *Conference on harmonic analysis in honor of Antoni Zygmund, Vol I, II* (Chicago, III, 1981), 590-606, *Wadsworth Math. Ser.*, Wadsworth, Belmont, CA, (1983)
- [18] G.B. Folland: A fundamental solution for a subelliptic operator, *Bull. Amer. Math. Soc.* 79, 373-376 (1973)
- [19] N. Garofalo, D.M. Nhieu: Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces, *Comm. Pure Appl. Math.*, **49**, 1081-1144 (1996)
- [20] N. Garofalo, D.M. Nhieu: Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot-Carathéodory spaces, *J. Anal. Math.*, **74**, 67-97 (1998)
- [21] A. Grigor'yan: Heat kernel and analysis on manifolds, *AMS/IP Studies in Advanced Mathematics*, 47. AMS, Providence, RI, International Press, Boston, MA, (2009)
- [22] P. Gyrya, L. Saloff-Coste: Neumann and Dirichlet heat kernels in inner uniform domains, *Asterisque* No. 336, viii+144 (2011)
- [23] R. Hamilton: A matrix Harnack estimate for the heat equation, *Comm. Anal. Geom.* 1, no.1, 113-126 (1993)

- [24] D. Jerison: The Poincaré inequality for vector fields satisfying Hörmander’s condition, Duke Math. J. 53, no. 2, 503-523 (1986)
- [25] D. Jerison, A. Sánchez-Calle: Subelliptic second order differential operators, Lectures. Notes in Math., 1277, 46-77 (1987)
- [26] B. Kotschwar: Hamilton’s gradient estimate for the heat kernel on complete manifolds, Proc. Amer. Math. Soc. 135, no.9, 3013-3019 (2007)
- [27] S. Kusuoka, D. Stroock: Applications of the Malliavin calculus III, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 34, no. 2, 391-442 (1987)
- [28] P. Lee, C. Li and I. Zelenko: Measure contraction properties of contact sub-Riemannian manifolds with symmetry, arXiv:1304.2658, preprint (2013)
- [29] P. Li, S.T. Yau: On the parabolic kernel of the Schrödinger operator, Acta Math. 156, 153-201 (1986)
- [30] P. Li: Uniqueness of  $L^1$  solutions for the Laplace equation and the heat equation on Riemannian manifolds, J. Diff. Geom. 20, no.2, 447-457 (1984)
- [31] J. Moser: On Harnack’s theorem for elliptic differential equations, Comm. Pure Appl. Math. 14, 577-591 (1961)
- [32] J. Moser: A Harnack inequality for parabolic differential equations, Comm. Pure Appl. Math. 17, 101-134 (1964)
- [33] A. Nagel, E.M. Stein, S. Wainger: Balls and metrics defined by vector fields. I. Basic properties, Acta Math. 155, no. 1-2, 103-147 (1985)
- [34] R. S. Phillips, L. Sarason: Elliptic-parabolic equations of the second order, J. Math. Mech. 17, 891-917 (1967/1968)
- [35] L. Saloff-Coste: Aspect of Sobolev-type inequalities, London Math. Soc. Lecture Notes Series 289. Cambridge University Press (2002)
- [36] L. Saloff-Coste: A note on Poincaré, Sobolev, and Harnack inequalities, Internat. Math. Res. Notices, no. 2, 27-38 (1992)
- [37] L. Saloff-Coste: Uniformly elliptic operators on Riemannian manifolds, J. Differential Geom. 36, no. 2, 417-450 (1992)
- [38] R. Strichartz, Sub-Riemannian geometry, J. Differential Geom. 24, no. 2, 221-263 (1986)
- [39] K.T. Sturm: Analysis on local Dirichlet spaces. I, J. Reine Angew. Math. 456, 173-196 (1994)

- [40] K.T. Sturm: Analysis on local Dirichlet spaces. II, Osaka J. Math. 32, no. 2, 275-312 (1995)
- [41] K.T. Sturm: Analysis on local Dirichlet spaces. III, J. Math. Pures Appl. (9) 75, no. 3, 273-297 (1996)
- [42] N. Varopoulos: Hardy-Littlewood theory for semigroups, J. Funct. Anal. 63, no. 2, 240-260 (1985)